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ORIGINAL PAPER

Solutions to linear matrix ordinary differential equations via minimal, regular, and excessive space extension based universalization

Perturbative matrix splines, convergence and error estimate issues for polynomial coefficients in the homogeneous case

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Abstract This work focuses on the solution of the linear matrix ordinary differential equations where the first derivative of the unknown matrix is equal to the same unknown matrix premultiplied by a given matrix polynomial of the independent variable as done in the previous paper. However, this time, not series but perturbation expansions are considered. Sufficient attention is given on the convergence and error estimate issues. The repetitious usage of the perturbation truncations on different but neighbor intervals permits us to define and use so-called "Perturbative Matrix Splines". This is somehow an analytic continuation issue. Certain illustrative applications are also presented to support the ideas of the work.

Keywords Ordinary differential equations · Space extension · Perturbation solutions · Majorant series · Error estimates

1 Introduction

One of our previous works, to which this paper somehow can be considered companion [1], has been designed to focus on the matrix ordinary differential equation (MODE) explicitly given below

$$\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t), \tag{1}$$

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where $\mathbf{X}(t)$ and $\mathbf{A}(t)$ are $n \times n$ type matrices, the former of which is the unknown of the problem and the latter one is the known matrix valued function of t, the independent variable which is assumed to be real valued so are the other entities of the problem. The given coefficient matrix $\mathbf{A}(t)$ is taken as a polynomial of t, whose explicit structure is given through t-independent matrix coefficients denoted by subindexed $\mathbf{A}$s as follows [2]

$$\mathbf{A}(t) \equiv \sum_{i=0}^{m} t^i \mathbf{A}_i . \tag{2}$$

The solution of (1) can be made unique by imposing an initial condition for the unknown as below

$$\mathbf{X}(0) = \mathbf{I}_n , \tag{3}$$

where $\mathbf{I}_n$ stands for the $n \times n$ type unit matrix. We could equivalently choose any constant matrix for $\mathbf{X}(0)$ to get highest level parametrization. However, this choice seems to be better since the solution for this initial condition can be postmultiplied by a constant matrix to get that matrix multiplicant as the initially imposed value for the unknown [3]. In our previous paper we have applied the space extension method to the above MODE by defining

$$\mathbf{X}(t) \equiv \sum_{i=0}^{m_{se}} t^i \mathbf{Y}_i(t), \tag{4}$$

where the positive integer $m_{se} + 1$ denotes the space extension order while $n \times n$ type temporal matrix function $\mathbf{Y}_i(t)$ is expressible as a Maclaurin series of not t but $t^{m_{se}+1}$. The right hand side of this equality is a linear combination of $m_{se} + 1$ linearly independent functions. Its insertion into (1) produces an expression which can be reorganized as the linear combination of $m_{se} + 1$ linearly independent expressions each of which must be individually vanish for the satisfaction of the total equation. These settings to zero matrix give $m_{se} + 1$ matrix equations which can be put into a single matrix equation as follows

$$\mathbf{Z}'(t) = \left[\frac{1}{t}\mathbf{B}_1 + t^{m_{se}} \mathbf{B}_2 \right] \mathbf{Z}(t), \tag{5}$$

where

$$\mathbf{Z}(t) \equiv \left[\mathbf{Y}_0(t)^T \ldots \mathbf{Y}_{m_{se}}(t)^T \right]^T \tag{6}$$

and

$$
\mathbf{B}_1 \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_0 & -\mathbf{I}_n & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{A}_0 & -2\mathbf{I}_n & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{A}_{m_{se}-1} & \cdots & \mathbf{A}_1 & \mathbf{A}_0 & -m_{se}\mathbf{I}_n \end{bmatrix} \tag{7}
$$

$$
\mathbf{B}_2 \equiv \begin{bmatrix} \mathbf{A}_{m_{se}} & \mathbf{A}_{m_{se}-1} & \cdots & \cdots & \mathbf{A}_0 \\ \mathbf{0} & \mathbf{A}_{m_{se}} & \mathbf{A}_{m_{se}-1} & \cdots & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{A}_{m_{se}} \end{bmatrix}. \tag{8}
$$

It is better to emphasize on m_{se} at this moment. It is one less than the space extension order and takes the value of m as its minimum. We called the case where $m_{se} = m$ "Minimal Space Extension" which does not reflect the more general structure of $\mathbf{B}_1$ [4,5]. All m_{ex} values greater than m are called "Excessive Space Extension". The minimal case of this expansion is, where m_{se} takes the value of $m + 1$ and $\mathbf{B}_1$ explicitly shows its more general structure of having all diagonals composed of separate **A**s. Hence this specific case is called "Regular Space Extension"[1]. The following series expansion can be written as a solution to (5)

$$
\mathbf{Z}(t) \equiv \sum_{i=0}^{\infty} t^{(m_{se}+1)i+\alpha} \mathbf{Z}_i, \tag{9}
$$

where the unknown scalar exponent α is chosen in such a way that the recursion amongst the coefficients of this expansion can be solved [3,6]. The unknown $\mathbf{Z}_i$ coefficients are of $(m + 1)n \times n$ matrices which are independent of t. As can be easily shown, the possible values of α are the eigenvalues of the matrix $\mathbf{B}_1$ and these values are all integers between $-m_{se}$ and 0 inclusive with the common multiplicity equal to n. It is also possible to show that the algebraic and geometric multiplicities of these eigenvalues are the same. Therefore n number of eigenvector can be directly found for each eigenvalues [7]. Each of these eigenvectors should be the value of $\mathbf{Z}_0$ for the α value corresponding to that eigenvector. Although the eigenvalue differences are all integer, there is no possibility of logarithmic solution since logarithmic series become a necessity only when any two eigenvalues have the difference equal to $m_{se} + 1$ [8]. This is not the case within the spectrum of $\mathbf{B}_1$. We have shown that the zero value of α is sufficient to go back to the solution of (1). To clearly recall this fact, we can use the above series expansion assumption in (5) to get the following two term recursion

$$
[\,((m+1)i+\alpha)\mathbf{I} - \mathbf{B}_1\,]\,\mathbf{Z}_i = \mathbf{B}_2 \mathbf{Z}_{i-1}, \quad i = 0, 1, 2, \ldots \tag{10}
$$

where $\mathbf{Z}_{-1}$ identically vanishes by convention. This recursion has the following formal solution

$$\mathbf{Z}_i = \left[\prod_{j=0}^{i-1} [\, ((m+1)(i-j)+\alpha)\mathbf{I} - \mathbf{B}_1\,]^{-1} \mathbf{B}_2 \right] \mathbf{Z}_0, \quad i = 1, 2, \ldots \tag{11}$$

which becomes unique only when $\mathbf{Z}_0$ is uniquely given. We can explicitly write

$$\mathbf{Z}_0 = \left[\, \mathbf{Z}_{0,0}^T \ldots \mathbf{Z}_{0,m}^T \,\right]^T, \tag{12}$$

and then consider the eigenequation of $\mathbf{B}_1$ for the case where α vanishes. This results in the following recursion

$$i\, \mathbf{Z}_{0,i} = \sum_{j=0}^{i-1} \mathbf{A}_{i-j-1} \mathbf{Z}_{0,j}, \quad i = 0, 1, \ldots, m, \quad \mathbf{Z}_{0,0} \equiv \mathbf{C}, \tag{13}$$

where $\mathbf{C}$ is arbitrary and we have used the vanishing convention for the sums with smaller upper bounds than the lower ones. This recursion matches with the first $m_{se}+1$ equations of the recursion amongst the matrix coefficients of the series solution to (1). The solutions of these two recursions' first $m_{se}+1$ equations also match when $\mathbf{Z}_{0,0}$ is taken as $\mathbf{I}_n$. Hence $n \times n$ matrix function $\mathbf{T}(t)^T \mathbf{Z}(t)$ for $\alpha = 0$ and

$$\mathbf{T}(t) \equiv \left[\, \mathbf{I}\; t\mathbf{I} \ldots t^m \mathbf{I} \,\right]^T \tag{14}$$

is the solution of (1) under the unit matrix initial condition. For this reason we deal with only the case where α vanishes. We have called the MODE in (5) "Universalized Form". We have investigated the convergence of the series expansion to the solution of (5) via a majorant series constructed with the aid of certain norm inequalities and have shown that the series converge in every finite disk centered around the origin in t-complex plane. This fact is in good harmony with the theory of ordinary differential equations as we expect and facilitating the construction of error estimates to truncation approximants. We defined the error estimates to the solution of (5) as follows

$$\mathbf{E}_i(t) \equiv \mathbf{Z}(t) - \mathbf{Z}_{T,i}(t) \equiv \sum_{j=i+1}^{\infty} t^{(m+1)j} \mathbf{Z}_j, \quad i = 0, 1, 2, \ldots \tag{15}$$

where $\mathbf{Z}_{T,i}(t)$ stands for the first $i+1$ terms of the series solution and we have dealt with the case of vanishing α as we emphasized that this is the most important and necessary item for our method. We find it sufficient to report the results of error construction procedure in integral representation form as follows [1]

$$\|\mathbf{E}_i(t)\| \le \|\mathbf{Z}_{i+1}\| \, \|\mathbf{B}_2\| \, t^{(m+1)i_{lb}} \int\limits_0^{t^{m+1}} d\tau \tau^m \left(t^{m+1} - \tau^{m+1}\right)^{i+1-i_{lb}} \mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1}\tau^{m+1}} + t^{(m+1)(i+1)} \|\mathbf{Z}_{i+1}\| , \quad i = i_{lb}, i_{lb}+1 \ldots \tag{16}$$

and in the following algebraic form

$$\|\mathbf{E}_i(t)\| \le \|\mathbf{Z}_{i+1}\| \frac{\|\mathbf{B}_2\|}{(m+1)} \frac{t^{(m+1)(i+2)}}{i+2-i_{lb}} \chi_{i+1-i_{lb}} \left(\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1} \right) + t^{(m+1)(i+1)} \|\mathbf{Z}_{i+1}\| , \qquad i = i_{lb}, i_{lb}+1 \ldots \tag{17}$$

where

$$\chi_i(t) \equiv \frac{(i+1)!}{t^{i+1}} \left(\mathrm{e}^t - \sum_{j=0}^{i} \frac{t^j}{j!} \right), \quad i = 0, 1, 2, \ldots \tag{18}$$

Paper is organized as follows. The second section involves the perturbation expansion for the solution of (5) at a nonzero value of the independent variable with a unit matrix initial condition at that point. The third section contains the truncation approximants, their convergences and error estimates while the fourth section is designed for the presentation of analytic continuation based procedure to connect different point perturbation expansions. There, we mention the possibility of the perturbative matrix spline definition and the utilization. The fifth section is devoted to illustrative examples. The sixth section finalizes the paper by giving concluding remarks.

2 Perturbation expansion at a nonzero point with an arbitrary matrix initialization

Our previous paper relevant to these topics [1], was for constructing a series solution at $t = 0$ and what we have shown via error estimates and validations by illustrative examples was the decrease in the numerical approximation quality as t gets away from $t = 0$. We have also shown that this negative tendency can be suppressed by using excessive space extension by working with sufficiently high orders. The other alternative to this suppression is to get an expansion not at $t = 0$ but some other point, say $t = t_1$, for (5). However, such an expansion results in not two but three consecutive term recursion and it is quite hard to find analytic solutions to that unless some very specific structures ap pear in the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$. Hence we do not intend to follow this route. On the other hand, it is always possible to artificially insert a perturbation parameter into (5) and then expand the unknowns into Maclaurin series of that parameter, whose coefficients satisfy a two term recursion. The resulting series should match with the solution of (5) when this perturbation parameter becomes 1 as long as the obtained perturbation series converge at 1 value of perturbation parameter. We will trace this path to obtain a solution to (5) under the initial condition which enforces the unknown matrix value $\mathbf{Z}(t_1)$ to be an arbitrary matrix of $(m_{se}+1)\,n \times n$ say $\mathbf{C}_1$ which will be later determined to relate the solutions at $t=0$ and $t=t_1$. Now we can consider a more general form of (5) as follows

$$\mathbf{Z}'(t,\varepsilon) = \left[\frac{1}{t_1}\mathbf{B}_1 + t_1^{m_{se}}\mathbf{B}_2\right]\mathbf{Z}_G(t,\varepsilon) + \varepsilon\left[\left(\frac{1}{t}-\frac{1}{t_1}\right)\mathbf{B}_1 + \left(t^{m_{se}} - t_1^{m_{se}}\right)\mathbf{B}_2\right]\mathbf{Z}_G(t,\varepsilon), \quad \mathbf{Z}_G(t_1,\varepsilon) = \mathbf{C}_1 \tag{19}$$

where we have considered the right hand side matrix function's value, not at a variant t value but specific point t_1, as the dominant term and its difference from the actual right hand side term as the perturbation. As long as it exists, the solution of the above problem is related to the solution of (5) via

$$\mathbf{Z}(t) = \mathbf{Z}_G(t,1). \tag{20}$$

To find a solution to (19) we can construct the following perturbation series by assuming that the solution is analytic in a disk centered at the origin with a nonzero radius in the complex plane of ε such that it involves $\varepsilon = 1$ [9]

$$\mathbf{Z}_G(t,t_1,\varepsilon) \equiv \sum_{i=0}^{\infty} \varepsilon^i \mathbf{Z}_{G,i}(t,t_1), \tag{21}$$

where t_1 dependence is deliberately shown. It is because of the fact that the point t_1 where accompanying initial condition is imposed affects the solution's behaviour at least in convergence domain. The insertion of (21) into (19) produces the following recursive MODE

$$\mathbf{Z}'_{G,i}(t,t_1) = \left[\frac{1}{t_1}\mathbf{B}_1 + t_1^{m_{se}}\mathbf{B}_2\right]\mathbf{Z}_{G,i}(t,t_1) + \left[\left(\frac{1}{t}-\frac{1}{t_1}\right)\mathbf{B}_1 + \left(t^{m_{se}} - t_1^{m_{se}}\right)\mathbf{B}_1\right]\mathbf{Z}_{G,i-1}(t,t_1), \quad \mathbf{Z}_{G,i}(t_1,t_1) = \mathbf{C}_1\delta_{i0}, \tag{22}$$

where subindexed δ stands for the Kronecker's delta symbol and $\mathbf{Z}_{-1}(t)$ is assumed to be identically vanishing. This equation takes the following form when $i=0$

$$\mathbf{Z}'_{G,0}(t,t_1) - \mathbf{C}\mathbf{Z}_{G,0}(t,t_1) = 0, \quad \mathbf{Z}_{G,0}(t_1,t_1) = \mathbf{C}_1, \tag{23}$$

where

$$\mathbf{C} \equiv \frac{1}{t_1}\mathbf{B}_1 + t_1^{m_{se}}\mathbf{B}_2. \tag{24}$$

The solution of (23) can be written as follows by skipping the intermediate construction details [3]

$$\mathbf{Z}_{G,0}\left(t,t_1\right)=\mathrm{e}^{(t-t_1)\mathbf{C}}\mathbf{C}_1. \tag{25}$$

On the other hand, by using $\mathbf{C}$ to avoid showing the matrix $\mathbf{B}_2$ explicitly the recursive MODE in (22) takes the following form

$$\mathbf{Z}'_{G,i}\left(t,t_1\right)-\mathbf{C}\mathbf{Z}_{G,i}\left(t,t_1\right)=\left[\left(\frac{1}{t}-\frac{t^{m_{se}}}{t_1^{m_{se}+1}}\right)\mathbf{B}_1 + \left(\frac{t^{m_{se}}}{t_1^{m_{se}}}-1\right)\mathbf{C}\right]\mathbf{Z}_{G,i-1}\left(t,t_1\right),\quad \mathbf{Z}_{G,i}\left(t_1,t_1\right)=\mathbf{0},\quad i=1,2,3,\ldots \tag{26}$$

whose solution for $\mathbf{Z}_{G,i}\left(t,t_1\right)$ in terms of others gives the following integral recursion

$$\mathbf{Z}_{G,i}\left(t,t_1\right)=\mathscr{I}\left(t_1\right)\mathbf{Z}_{G,i-1}\left(t,t_1\right)\equiv\int_t^{t_1}d\tau\mathbf{K}\left(t,\tau\right)\mathbf{Z}_{G,i-1}\left(\tau,t_1\right),\quad i=1,2,3,\ldots \tag{27}$$

where

$$\mathbf{K}\left(t,\tau\right)\equiv\left(1-\frac{\tau^{m_{se}}}{t_1^{m_{se}}}\right)\mathbf{C}\mathrm{e}^{t\mathbf{C}}-\left(\frac{1}{\tau}-\frac{\tau^{m_{se}}}{t_1^{m_{se}+1}}\right)\mathrm{e}^{(t-\tau)\mathbf{C}}\mathbf{B}_1\mathrm{e}^{\tau\mathbf{C}}. \tag{28}$$

(27) makes $\mathbf{Z}_{G,i}\left(t,t_1\right)$ the image of $\mathbf{Z}_{G,i-1}\left(t,t_1\right)$ under the integral operator $\mathscr{I}\left(t_1\right)$ which has a matrix kernel ($\mathbf{K}\left(t,\tau\right)$). Since this integral operator does not depend on i it is not wrong to consider that $\mathbf{Z}_{G,i}\left(t,t_1\right)$ can be interpreted as the image of $\mathbf{Z}_0\left(t,t_1\right)$ under the ith power of same operator. Thus we can write

$$\mathbf{Z}_{G,i}\left(t,t_1\right)=\mathscr{I}\left(t_1\right)^i\mathbf{Z}_{G,0}\left(t,t_1\right)=\mathscr{I}\left(t_1\right)^i\mathrm{e}^{(t-t_1)\mathbf{C}}\mathbf{C}_1,\qquad i=1,2,3,\ldots \tag{29}$$

This facilitates a bound analysis to get information about the convergence of the perturbation series. It is better to work in a closed interval to facilitate our bound analysis. Hence we consider the t values belonging to the interval $[\,t_0,t_1\,]$ where t_0 is a positive value for our error estimate studies here. Now we can define the following truncation approximants

$$\mathbf{Z}_G^{(i)}\left(t,t_1,\varepsilon\right)=\sum_{j=0}^{i}\varepsilon^j\mathbf{Z}_{G,j}\left(t,t_1\right),\qquad i=1,2,3,\ldots \tag{30}$$

whose value for $\varepsilon = 1$ defines the ith perturbation approximant to the solution of (5) as follows

$$\mathbf{Z}^{(i)}(t, t_1) = \mathbf{Z}_G^{(i)}(t, t_1, 1), \qquad i = 1, 2, 3, \ldots \tag{31}$$

3 Convergence and error estimates for truncations approximants

The matrices in the kernel of $\mathscr{I}(t_1)$ given in (28) continuously depend on their arguments. Hence they are bounded in $[\, t_0, t_1\,]$. This implies that we can find maximum values for them throughout this interval and therefore we can define

$$\kappa_1(t_0, t_1) \equiv \max_{t\in[\,t_0,t_1\,]} \left\| \mathrm{e}^{(t-\tau)\mathbf{C}} \mathbf{B}_1 \mathrm{e}^{\tau\mathbf{C}} \right\|, \qquad \kappa_2(t_0, t_1) \equiv \max_{t\in[\,t_0,t_1\,]} \left\| \mathbf{C}\mathrm{e}^{t\mathbf{C}} \right\| \tag{32}$$

and write

$$\left(\frac{1}{\tau} - \frac{\tau^{m_{se}}}{t_1^{m_{se}+1}}\right) < \left(\frac{1}{t_0} - \frac{t_0^{m_{se}}}{t_1^{m_{se}+1}}\right), \qquad t \in [\, t_0, t_1\,] \tag{33}$$

and

$$\left(1 - \frac{\tau^{m_{se}}}{t_1^{m_{se}}}\right) < \left(1 - \frac{t_0^{m_{se}}}{t_1^{m_{se}}}\right), \qquad t \in [\, t_0, t_1\,]. \tag{34}$$

All these urge us to define

$$\kappa(t_0, t_1) \equiv \left(\frac{1}{t_0} - \frac{t_0^{m_{se}}}{t_1^{m_{se}+1}}\right)\kappa_1(t_0, t_1) + \left(1 - \frac{t_0^{m_{se}}}{t_1^{m_{se}}}\right)\kappa_2(t_0, t_1), \tag{35}$$

and therefore to get the following inequality from (27)

$$\left\| \mathbf{Z}_{G,i}(t, t_1) \right\| < \kappa(t_0, t_1) \int_t^{t_1} d\tau \left\| \mathbf{Z}_{G,i-1}(\tau, t_1) \right\|, \qquad i = 1, 2, 3, \ldots \tag{36}$$

If we write the following inequality from (25)

$$\left\| \mathbf{Z}_{G,0}(t, t_1) \right\| = \mathrm{e}^{(t_1-t_0)\|\mathbf{C}\|} \left\| \mathbf{C}_1 \right\|, \tag{37}$$

then we can conclude

$$\left\| \mathbf{Z}_{G,i}(t, t_1) \right\| < \frac{\kappa(t_0, t_1)^i}{i!} (t_1 - t)^i \, \mathrm{e}^{(t_1-t_0)\|\mathbf{C}\|} \left\| \mathbf{C}_1 \right\|, \qquad i = 0, 1, 2, \ldots \tag{38}$$

by sufficient number of consecutive application of (36). Now from (21) we can write [10]

$$\left\| \mathbf{Z}_G\left(t, t_1, \varepsilon\right) \right\| \leq \sum_{i=0}^{\infty} |\varepsilon|^i \left\| \mathbf{Z}_{G,i}\left(t, t_1\right) \right\|, \tag{39}$$

which can be combined with (38) to get

$$\left\| \mathbf{Z}_G\left(t, t_1, \varepsilon\right) \right\| \leq \mathrm{e}^{\kappa\left(t_0, t_1\right)|\varepsilon|\left(t_1 - t\right)} \mathrm{e}^{\left(t_1 - t_0\right)\|\mathbf{C}\|} \left\| \mathbf{C}_1 \right\|. \tag{40}$$

This means that the perturbation expansion above converges for all finite values of perturbation parameter as long as $\kappa\left(t_0, t_1\right)$ remains finite. This is provided by the positive values of t_0. How ever, $\kappa\left(t_0, t_1\right)$ grows unboundedly as t_0 gets closer to zero. Hence, although there is a conceptual convergence, the convergence rate gets down as t_0 diminishes. This implies that the perturbation expansion above becomes fruitfull as t_0 is sufficiently far from zero. We can focus on the error estimates for the truncation errors. We can define

$$E_i\left(t, t_1\right) \equiv \left\| \mathbf{Z}_G\left(t, t_1, 1\right) - \mathbf{Z}_G^{(i)}\left(t, t_1, 1\right) \right\|, \qquad i = 1, 2, 3, \ldots \tag{41}$$

and write

$$E_i\left(t, t_1\right) \leq \sum_{j=i+1}^{\infty} \left\| \mathbf{Z}_{G,j}\left(t, t_1\right) \right\|, \tag{42}$$

which implies

$$E_i\left(t, t_1\right) \leq \sum_{j=i+1}^{\infty} \frac{\kappa\left(t_0, t_1\right)^j}{j!} \left(t_1 - t\right)^j \mathrm{e}^{\left(t_1 - t_0\right)\|\mathbf{C}\|} \left\| \mathbf{C}_1 \right\|, \tag{43}$$

or in terms of the function χ_i given by (18)

$$E_i\left(t, t_1\right) \leq \frac{\kappa\left(t_0, t_1\right)^{i+1}}{(i+1)!} \left(t_1 - t\right)^{i+1} \chi_i\left(\kappa\left(t_0, t_1\right)\left(t_1 - t\right)\right) \mathrm{e}^{\left(t_1 - t_0\right)\|\mathbf{C}\|} \left\| \mathbf{C}_1 \right\|. \tag{44}$$

This is the error estimate for the ith truncation approximant of the perturbation expansion under consideration.

4 Analytic continuation based initial constant determination

As we know from the above analysis $\mathbf{Z}_G\left(t, t_1\right)$ has an arbitrary constant $\mathbf{C}_1$ at right. To facilitate its determination we write

$$\mathbf{Z}\left(t, t_1\right) \equiv \overline{\mathbf{Z}}\left(t, t_1\right) \mathbf{C}_1, \tag{45}$$

where the matrix valued function $\overline{\mathbf{Z}}(t, t_1)$ now is a unique entity. If we consider another solution at a different t value, say t_2, by writing

$$\mathbf{Z}(t, t_2) \equiv \overline{\mathbf{Z}}(t, t_2)\,\mathbf{C}_2, \tag{46}$$

where $\overline{\mathbf{Z}}(t, t_2)$ stands for the unique matrix valued function evaluated around $t = t_2$ and $\mathbf{C}_2$ denotes a new arbitrary matrix. If we assume $t_2 > t_1$ then the values of the matrix functions given in (45) and (46) should match at intermediate point like the mid point in the interval $[t_1, t_2]$ since the uniqueness of the MODE's solution implies their linear dependence. Hence we can write

$$\overline{\mathbf{Z}}\left(\frac{1}{2}(t_1 + t_2), t_2\right)\mathbf{C}_2 = \overline{\mathbf{Z}}\left(\frac{1}{2}(t_1 + t_2), t_1\right)\mathbf{C}_1 \tag{47}$$

from where one of the constant matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ can be solved in terms of the other as long as the matrices $\overline{\mathbf{Z}}\left(\frac{1}{2}(t_1 + t_2), t_2\right)$ and $\overline{\mathbf{Z}}\left(\frac{1}{2}(t_1 + t_2), t_1\right)$ permit. If this does not happen then the matching point can be changed from midpoint to some other interior point of the interval $[t_1, t_2]$. The matching point issue may be more comprehensive than it stands and may require more detailed analyses. However, it is out of the scope of this work. We assume that we can find a nonproblematic matching point and use it in our practical applications. These issues here bring the "Perturbative Matrix Spline" concept into mind. Assume that we have three perturbative expansions at the points where t takes the increasing values t_1, t_2, and, t_3 and the solutions take the constant matrix values $\mathbf{C}_1$, $\mathbf{C}_2$, and, $\mathbf{C}_3$ at these points, respectively. We denote the matching points by $\overline{t}_{1-2}$ and $\overline{t}_{2-3}$ for the first and second, and, second and third perturbative solutions. Then, the second solution can be considered as the representation of a piecewise matrix function over the interval $\left[\overline{t}_{1-2}, \overline{t}_{2-3}\right]$. By making an analogy to the spline functions of numerical analysis [11], the totality of these type matrix functions can be called "Perturbative Matrix Splines". The first interval will have 0 as the beginning point and to the first matching point the perturbative matrix spline is represented by the truncated series solution around zero. The truncation points and therefore spline intervals may vary depending on the truncation order in the perturbation series and the series solution truncation order.

5 Illustrative implementations

We consider the case where $\mathbf{A}(t)$ is taken as follows

$$\mathbf{A}(t) \equiv \begin{bmatrix} -202t - 117 & -294t - 168 \\ 140t + 80 & 204t + 115 \end{bmatrix}, \tag{48}$$

which was also considered in the previous paper. The analytic solution to (1) for this specific case can be explicitly given as follows

$$\mathbf{X}(t) \equiv \begin{bmatrix} 15e^{-3t^2-5t} - 14e^{4t^2+3t} & 21e^{-3t^2-5t} - 21e^{4t^2+3t} \\ 10e^{4t^2+3t} - 10e^{-3t^2-5t} & 15e^{4t^2+3t} - 14e^{-3t^2-5t} \end{bmatrix}. \tag{49}$$

The explicit structures of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$, and, $\mathbf{Z}_0$ are given below

$$\mathbf{B}_1 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 117 & -168 & -1 & 0 \\ 80 & 115 & 0 & -1 \end{bmatrix}, \quad \mathbf{B}_2 \equiv \begin{bmatrix} -202 & -294 & -117 & -168 \\ 140 & 204 & 80 & 115 \\ 0 & 0 & -202 & -294 \\ 0 & 0 & 140 & 204 \end{bmatrix} \tag{50}$$

$$\mathbf{Z}_0 \equiv \begin{bmatrix} 1 & 0 & -117 & 80 \\ 0 & 1 & -168 & 115 \end{bmatrix}^T. \tag{51}$$

Five curves for the Frobenius norms of various matrix entities are depicted in Fig. 1 for the interval [0.5, 1.0]. The exact solution norm is given by the second curve from above and colored by red. The first, third, and fourth curves from above are the single term perturbative solutions expanded at $t = 1.0$. They are matched the four term series solution truncation at $t = 0.70$, $t = 0.78$, and $t = 0.85$, respectively. They are painted in orange, black, and blue colors downwardly. The bottommost curve is corresponding to the five term truncated series solution and is colored in brick. As can be noticed immediately the third curve from above almost same as the exact solution throughout the interval. In fact, its deviation from the exact solution is deliberately chosen sufficiently large to be able to show two curves without screening one of which by the other. The true matching point less than 0.78 although we have not attempted to evaluate it precisely. This precise calculation may require to construct an optimisation problem which is considered beyond the scope of this paper. The overcoming quality of the single term perturbative matrix spline to the five first term truncation of the series solution is apparent. The curves in Fig. 2 are designed for the case where the perturbation expansion is at $t = 1.5$. The second curve is again for the exact solution while the first, third, and fourth curves are corresponding to the norms of the single term perturbative matrices matched with the perturbative matrix of Fig. 1 at the points $t = 1.20$, $t = 1.28$, $t = 1.36$, respectively. The third (black) curve is the best approximation in this figure. However, it is not the best possible one. It is deliberately designed to be noticably deviating from the exact solution for technical reasons. The bottommost curve this time is the best single term perturbative matrix of Fig. 2. The quality evidently increases by using splines again.

6 Concluding remarks

In the first one of these two papers [1], we developed a new method to solve linear matrix ordinary differential equations via space extension. We obtained an universal

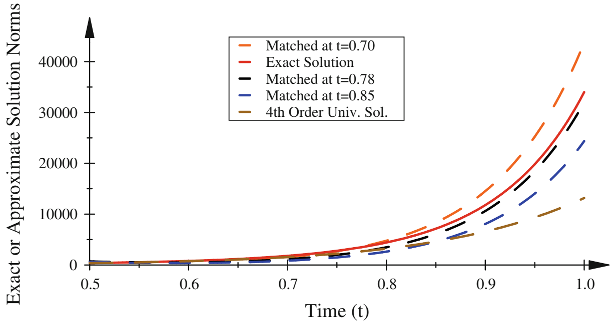

Fig. 1 The comparison of the Frobenius norms of the exact solution and single term perturbative solution for the interval [0.5, 1]

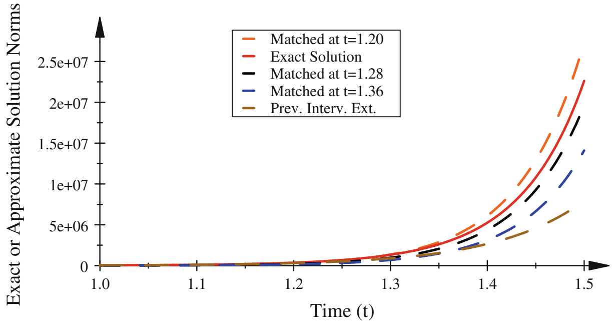

Fig. 2 The comparison of the Frobenius norms of the exact solution and single term perturbative solution for the interval [1.0, 1.5]

form for these equations by using space extension concept. We developed different types of space extension expressions with respect to the order of approximations. We showed that the quality of approximation increases as the order of space extension grows. However, we also observed that truncation approximation quality tends to decrease as t goes away from 0. This drawback urged us to use some other series expansions at some other points which are sufficiently far from 0. The solution based on space extension concept generates a two term recursion but if we use series expansion on the other points which are far from 0, this time the form obtained from series solution will not be a two term recursion. A three term recursion will be obtained and because of the fact that it is not straightforward to obtain an analytical solution of that recursion, we do not use series expansion, but this time we use perturbation expansion. Thus, it enables us to empower the method constructed in these two companion papers. In this paper, a perturbation parameter is introduced into the matrix ordinary differential equation and the equation is expanded into Maclaurin series whose coefficients satisfy a two term recursion and thus a new truncation approximation is constructed. We also investigated the convergence and error estimates for these truncation approximants.

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